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B-Splines, Pólya Curves, and Duality

PHILLIP J. BARRY

Computer Science Deparment, University of Minnesota, 4-192 EE/CSci Building, 200 Union St. SE, Minneapolis, Minnesota 55455

RONALD N. GOLDMAN

Department of Computer Science, Rice University, Houston, Texas 77251

AND

TONY D. DEROSE

Department of Computer Science, FR-35, University of Washington, Seattle, Washington 98195

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Local duality between B-splines and Pólya curves is examined, mostly from the viewpoint of computer-aided geometric design. Certain known results for the two curve types are shown to be related. A few new results for Pólya curves and a curve scheme related to B-splines also follow from these investigations.

1. INTRODUCTION

A powerful technique for investigating the properties of B-splines is to exploit the properties of the functions $\psi_i(t) = \prod_{j=i+1}^{i+n} (t_j - t)$ (where the t_j are knots and the spline is of degree n (order $n+1$)) which bear an intimate relationship to B-splines (see, e.g., [7]). These functions are also quite similar to the blending functions of the Pólya curves presented in [10] and further developed in [1, 2]. Pólya curves are polynomial generalizations of Bézier curves, and share many of the features which make Bézier curves suitable for CAGD (computer-aided geometric design). Because of the similarity of the Pólya curve blending functions to the functions $\psi_i(t)$, there is an intimate relationship between B-splines and Pólya curves. The main purpose of this paper is to examine this relationship from

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the viewpoint of CAGD, and note the connection between certain results for B-spline curves and certain results for Pólya curves.

The paper is structured as follows: in Section 2 we set some notation, explain our approach, and review some known results. Section 3 begins the discussion of the relationship between B-splines and Pólya curves, and Section 4 continues this discussion by examining change of basis procedures. In Section 5 we employ some of the results from Section 4 to generate a few new results for Pólya curves and another curve scheme which is related to B-spline curves. Section 6 contains concluding remarks.

2. Approach, Notation, and Basic Results

Pólya curves are polynomial, B-spline curves are piecewise polynomial. To achieve greater compatibility between these two schemes we will deal with B-splines *locally*—that is, over one knot interval. Some of the results we obtain extend readily to results about B-spline curves; indeed certain of the results are modifications of results given, using the ψ_i, in [7, 12]. However, this paper differs from those in that our primary intent is not to derive properties of B-spline curves but to note the interplay between features of B-spline curve segments and features of Pólya curves.

For a given knot sequence $\mathbf{t}$, B-spline basis functions can be defined recursively by

$$\begin{aligned} b_i^0(x) &= 1 \qquad x \in [t_i, t_{i+1}) \\ &= 0 \qquad \text{else} \end{aligned}$$

$$b_i^n(x) = \frac{x - t_i}{t_{i+n} - t_i}\, b_i^{n-1}(x) + \frac{t_{i+n+1} - x}{t_{i+n+1} - t_{i+1}}\, b_{i+1}^{n-1}(x), \tag{1}$$

where the superscript denotes the degree and the subscript the leftmost possible point of support (i.e., $b_i^n(t)$ vanishes outside of $[t_i, t_{i+n+1})$). An nth degree B-spline curve can be written as

$$\mathbf{b}(x) = \sum b_i^n(x)\, P_i, \tag{2}$$

where the P_i are control vertices.

The B-spline curve segment over the single interval $[t_q, t_{q+1})$ is then

$$\mathbf{b}(x)|_{[t_q, t_{q+1})} = \sum_{i=q-n}^{q} (b_i^n(x)\, P_i)|_{[t_q, t_{q-1})}, \tag{3}$$

since the other basis functions vanish over $[t_q, t_{q+1})$. Therefore $\mathbf{b}(x)|_{[t_q, t_{q+1})}$ depends only on the $2n$ knots $t_{q-n+1}, \dots, t_{q+n}$.

Before Pólya curves are introduced, this setup must be generalized somewhat since Pólya curves can be defined by a knot sequence which is not necessarily nondecreasing (as long as, in our case, $t_{q-i} - t_{q+1+j} \neq 0$ for all $i, j \geqslant 0$, $i + j < n$, in order to avoid singularities), and can also be defined over all the reals. Since the *B*-splines are considered locally, it is possible to modify the above definitions to accommodate these generalizations. The functions in (1) are redefined by

$$b_i^0(x) \equiv \delta_{qi}$$

$$b_i^n(x) = \frac{x - t_i}{t_{i+n} - t_i} b_i^{n-1}(x) + \frac{t_{i+n+1} - x}{t_{i+n+1} - t_{i+1}} b_{i+1}^{n-1}(x) \qquad q - n \leqslant i \leqslant q \quad (4)$$

$$= 0 \qquad \text{else,}$$

over all the reals, and used in (2) to get what we will call the "generalized *B*-spline segment" $\mathbf{b}_q(x)$. Note that although (4) differs slightly from (1), over $[t_q, t_{q+1})$ they are equivalent in the usual *B*-spline context (nondecreasing knot sequence).

The Pólya blending functions have a recursion formula similar to Eq. (4). Define the blending functions by

$$d_i^0(t) \equiv \delta_{qi}$$

$$d_i^n(t) = \frac{t_{i+n} - t}{t_{i+n} - t_i} d_i^{n-1}(t) + \frac{t - t_{i+1}}{t_{i+n+1} - t_{i+1}} d_{i+1}^{n-1}(t) \qquad q - n \leqslant i \leqslant q \quad (5)$$

$$= 0 \qquad \text{else}$$

and the nth degree Pólya curve by

$$\mathbf{d}_q(t) = \sum_{i=q-n}^{q} d_i^n(t)\, P_i. \qquad (6)$$

Note that although both the Pólya and the generalized *B*-spline curve segment basis functions depend on q, this dependence is not represented in the notation. Also, Eqs. (5) and (6) differ from the presentation of Pólya curves given in [1, 2] where shape parameters μ_i and ν_i are used instead of knots; however, the two forms are related by

$$\mu_i = -t_{q-i}, \qquad \nu_i = t_{q+i+1} - 1. \qquad (7)$$

Moreover, the basis functions are labeled here differently, with the function $d_i^n(t)$ in [1, 2] corresponding to $d_{q-i}^n(t)$ of this paper.

Since the functions $d_i^n(t)$ and $b_i^n(x)$ are 0 if $i < q - n$ or $i > q$, when we mention the set of functions $d_i^n(t)$ or the set of functions $b_i^n(x)$ below we mean only those for which $q-n \leqslant i \leqslant q$.

Both the generalized B-spline and the Pólya blending functions have natural probabilistic interpretations and can be generated from simple stochastic models. Consider an urn which initially contains w white and b black balls. One ball at a time is now drawn at random from the urn, and its color inspected; then the ball is returned to the urn. Additional balls are then added to the urn in the following manner.

Pólya model: If the ball was the jth white ball [jth black ball] to be chosen, then $c_j(w+b)$ additional white balls [$d_j(w+b)$ additional black balls] are added to the urn.

B-spline model: If the ball was the jth white ball [jth black ball] to the chosen, then $d_j(w+b)$ additional black [$c_j(w+b)$ additional white balls] are added to the urn.

Let

$$x = t = \frac{w}{w+b}$$

$$c_i = \frac{t_{q\;\;i+1} - t_{q\;\;i}}{t_{q+1} - t_q} \qquad d_i = \frac{t_{q+i+1} - t_{q-i}}{t_{q+1} - t_q} \qquad i = 1, \ldots, n-1 \tag{8}$$

$\tilde{d}^n_{q\;\;i}(t) =$ the probability of selecting exactly i white balls in the first n trials from the Pólya urn model

$\tilde{b}^n_{q\;\;n+i}(x) =$ the probability of selecting exactly i white balls in the first n trials from the B-spline urn model.

Then it is easy to show probabilistically that the functions $\tilde{d}_i^n((t-t_q)/(t_{q+1}-t_q)$ and $\tilde{b}_i^n(x-t_q)/(t_{q+1}-t_q))$ satisfy (5) and (4), respectively. Therefore they are the B-spline and Pólya blending functions. This construction can be generalized further by permitting negative values for c_j, d_j, and letting x and t range over all the reals. Further details are given in [1, 10, 11].

These urn models for B-spline and Pólya blending functions are related in an obvious manner: in the Pólya model only balls of the same color as the color chosen are added to the urn; in the B-spline model only balls of the opposite color to the color chosen are added to the urn. One would therefore expect some interesting relationships between these probability distributions. These relationships are the major theme of this paper.

As mentioned above, Pólya curves share many features with Bézier curves. A few features we shall use in this paper are derived in [1, 2] and are listed below:

(1) Sum to unity,

$$\sum_{i=q\;\;n}^{q} d_i^n(t) = 1. \tag{9}$$

(2) (Nearly) explicit formula,

$$d_i^n(t) = \zeta_{n,i} \prod_{j=i+1}^{i+n} (t_j - t), \tag{10}$$

where the $\zeta_{n,i}$ are constants which can be found recursively.

(3) Basis. The $d_i^n(t)$ form a basis for the space of all degree n (or less) polynomials iff $\zeta_{n,i} \neq 0$ for $i = q-n, ..., q$. The functions $d_i^n(t)/\zeta_{n,i}$ will always form a basis, even if one or more of the $\zeta_{n,i} = 0$.

(4) Degree elevation,

$$d_i^n(t) = \frac{\zeta_{n,i}}{t_{i+n+1} - t_i} \left[\frac{d_i^{n+1}(t)}{\zeta_{n+1,i}} - \frac{d_{i-1}^{n+1}(t)}{\zeta_{n+1,i-1}} \right]. \tag{11}$$

(5) Interpolation at "endpoints,"

$$\mathbf{d}_q(t_q) = P_q \qquad \text{and} \qquad \mathbf{d}_q(t_{q+1}) = P_{q-n}. \tag{12}$$

(6) Reparametrization. Let $\hat{d}_i^n(t)$ denote the Pólya curve blending functions defined by the knot vector

$$(\hat{t}_{q-n+1}, ..., \hat{t}_{q+n}) = \left(\frac{t_{q-n+1} - c}{a}, \frac{t_{q-n+2} - c}{a}, ..., \frac{t_q - c}{a}, \frac{t_{q+1} - c}{a}, ..., \frac{t_{q+n} - c}{a} \right).$$

Then

$$d_i^n(at + c) = \hat{d}_i^n(t). \tag{13}$$

3. *B*-Splines, Pólya Curves, and Duality

We are now ready to examine the relationship between generalized *B*-spline segments and Pólya curves. The main tool will be the de Boor–Fix form of the dual basis for *B*-splines. Since a slightly modified case is studied, a proof of this result is included. It is then used to establish some relationships between the two curve schemes. A few preliminary lemmas are needed.

Lemma 1 (Marsden [13]).

$$(x - t)^n = \sum_{i=q-n}^{q} \frac{b_i^n(x)\, d_i^n(t)}{\zeta_{n,i}}. \tag{14}$$

Note that the inclusion of the normalizing factors into the $d_i^n(t)$ necessitates the denominator. The case where $\zeta_{n,i}=0$ (which may occur—see [1, 2]) presents no problem since $\zeta_{n,i}$ also appears in the numerator in $d_i^n(t)$.

Proof. By induction on n. The result is certainly true for $n=0$. Observe that for any i

$$(x-t)=\frac{(x-t_i)(t_{i+n}-t)}{t_{i+n}-t_i}+\frac{(t_{i+n}-x)(t_i-t)}{t_{i+n}-t_i}. \tag{15}$$

By the inductive hypothesis

$$\begin{aligned}(x-t)^n &= \sum_{i=q-n+1}^{q} \frac{(x-t)\, b_i^{n-1}(x)\, d_i^{n-1}(t)}{\zeta_{n-1,i}} \\ &= \sum \frac{(x-t_i)}{t_{i+n}-t_i} b_i^{n-1}(x) \frac{(t_{i+n}-t)}{\zeta_{n-1,i}} d_i^{n-1}(t) \\ &\quad + \sum \frac{(t_{i+n}-x)}{t_{i+n}-t_i} b_i^{n-1}(x) \frac{(t_i-t)}{\zeta_{n-1,i}} d_i^{n-1}(t).\end{aligned}$$

This equals, by use of (10),

$$\begin{aligned}&\sum \frac{(x-t_i)}{t_{i+n}-t_i} b_i^{n-1}(x) \frac{1}{\zeta_{n,i}} d_i^n(t) + \sum \frac{(t_{i+n}-x)}{t_{i+n}-t_i} b_i^{n-1}(x) \frac{1}{\zeta_{n,i-1}} d_{i-1}^n(t) \\ &= \sum \left[\frac{x-t_i}{t_{i+n}-t_i} b_i^{n-1}(x) + \frac{t_{i+n+1}-x}{t_{i+n+1}-t_{i+1}} b_{i+1}^{n-1}(x)\right] \frac{d_i^n(t)}{\zeta_{n,i}} \\ &= \sum_{i=q-n}^{q} \frac{b_i^n(x)\, d_i^n(t)}{\zeta_{n,i}}\end{aligned}$$

by (4). Q.E.D.

Define linear functionals λ_i^n (for $i=q-n, ..., q$) by

$$\lambda_i^n[f(x)] = \sum_{r=0}^{n} \frac{(-1)^{n-r}\, [d_i^n(\tau)]^{(n-r)}\, [f(\tau)]^{(r)}}{n!\, \zeta_{n,i}}, \tag{16}$$

where the value τ at which we evaluate the derivatives can be any real number.

That these are indeed functionals on the space of all polynomials of degree n or less follows from the next lemma.

Lemma 2 (see [5, p. 127]). *If f is a polynomial of degree less than or equal to n, then $\lambda_i^n f$ is a constant.*

Proof. This follows from the easily verified fact that the derivative of $[\lambda_i^n f]$ with respect to τ is identically 0.

LEMMA 3. $\lambda_i^n(x-t)^n = d_i^n(t)/\zeta_{n,i}$

Proof. In (16)$[(x-t)]^{(r)}|_{x=\tau}$ may be evaluated at any *hoice of* τ; choosing $\tau = t$ reduces (16) to $d_i^n(t)/\zeta_{n,i}$.

THEOREM 1 (de Boor and Fix [6]). $\lambda_i^n b_j^n = \delta_{ij}$.

Proof. By Lemma 1 $(x-t)^n = \sum_{j=q-n}^{q} d_j^n(t)\, b_j^n(x)/\zeta_{n,j}$. Apply λ_i^n to both sides and use Lemma 3 to get

$$\frac{d_i^n(t)}{\zeta_{n,i}} = \sum_{j=q-n}^{q} \frac{d_j^n(t)}{\zeta_{n,j}} \lambda_i^n b_j^n. \tag{17}$$

The result now follows from the linear independence of the functions $d_j^n(t)/\zeta_{n,j}$. Q.E.D.

Note that Theorem 1 implies that the $b_i^n(x)$ are always linearly independent. (Apply λ_j^n to both sides of $\sum_{i=q-n}^{q} c_i b_i^n(x) = 0$ to get $c_j = 0$ for every j.) Thus they form a basis and the λ_j^n comprise the dual basis.

Theorem 1 will be used to show how the possession of a desirable feature by generalized *B*-spline segments affects Pólya curves, and vice versa. We begin with some simple properties, first noting a relationship between the basis functions' lead coefficients, then mentioning a condition for nondegeneracy of Pólya curves, and finally looking at a few interpolation results.

First note, by differentiation of (10), that $[d_i^n(t)]^{(n)} = n!(-1)^n \zeta_{n,i}$. Furthermore, by differentiating (14) n times with respect to x, using (9) and the fact that the functions $\{d_i^n(t)/\zeta_{n,i}\}$ are linearly independent, one obtains $[b_i^n(x)]^{(n)} = n!\zeta_{n,i} = (-1)^n [d_i^n(t)]^{(n)}$. Since we can therefore replace the constants $\zeta_{n,i}$ in Marsden's identity (14) or the de Boor–Fix formula (16) by either $[b_i^n(x)]^{(n)}/n!$ or $(-1)^n [d_i^n(t)]^{(n)}/n!$, these formulas are (save for a factor of $(-1)^n$) symmetric in $d_i^n(t)$ and $b_i^n(x)$.

A curve scheme in CAGD is said to be nondegenerate if the only time the curve collapses to a single point is when all the control vertices are located at that point. A necessary and sufficient condition for a curve scheme to be nondegenerate is that its blending functions be linearly independent [10]. Pólya curves are thus nondegenerate iff no $\zeta_{n,i} = 0$. From the preceding remarks it follows that

THEOREM 2. *Over a knot vector* **t**, nth *degree Pólya curves are nondegenerate iff the blending functions of the corresponding generalized B-spline segment are all of exact degree n.*

One aspect of Pólya curves which makes them noteworthy is their interpolatory properties [1, 2]. Suppose a nondegenerate Pólya curve always interpolates P_j at $\hat{t}$. Then $d_i^n(\hat{t}) = \delta_{ij}$.

THEOREM 3. *A nondegenerate Pólya curve of degree n over a knot vector* $\mathbf{t}$ *interpolates* P_j *at* $\hat{t}$ *iff the generalized B-spline segment blending function* $b_j^n(x) = \zeta_{n,j}(x - \hat{t})^n$.

Proof. By Theorem 1 and Lemma 3 $b_j^n(x) = \zeta_{n,j}(x - \hat{t})^n$ iff $\delta_{ij} = \lambda_i^n[\zeta_{n,j}(x - \hat{t})^n]$ iff $\delta_{ij} = \zeta_{n,j} d_i^n(\hat{t})/\zeta_{n,i}$ iff the Pólya curve interpolates P_j at $\hat{t}$. Q.E.D.

Given values $s_0, ..., s_n$, if we choose the knots $t_{q+1-j} = s_{j-1}$ and $t_{q+j} = s_{n-j+1}$ for $j = 1, ..., n$, the Pólya blending functions become the Lagrange cardinal functions [1, 2]. We therefore get the following corollary.

COROLLARY. *A Pólya curve is the Lagrange interpolating curve iff the generalized B-spline segment blending functions* $b_i^n(x) = \zeta_{n,i}(x - s_{i+n-q})^n$.

As with *B*-splines, a generalized *B*-spline segment will interpolate at t_q or t_{q+1} to P_{q-n} or P_q if $t_{q-n+1} = t_{q-n+2} = \cdots = t_q$ or $t_{q+1} = t_{q+2} = \cdots = t_{q+n}$, respectively. This gives us the following result.

THEOREM 4. $\mathbf{b}_q(t_q) = P_{q-n}$ *iff* $d_{q-n}^n(t) = \zeta_{n,q-n}(t_q - t)^n$. $\mathbf{b}_q(t_{q+1}) = P_q$ *iff* $d_q^n(t) = \zeta_{n,q}(t_{q+1} - t)^n$.

4. CHANGE OF BASIS PROCEDURES

We now examine change of basis. Here duality has many interesting manifestations.

To begin, consider a simple change of basis to the generalized *B*-spline segment blending functions from some set of functions $g_i^n(x)$ $i = q - n, ..., q$ which form a basis for degree n or less polynomials. Then there exists a matrix B such that $b_j^n(x) = \sum_{i=q-n}^{q} B_{ij} g_i^n(x)$. In fact, it will be useful to look at a more general case involving derivatives of the $b_j^n(x)$. Consider the matrix B such that

$$[b_j^{n+s}(x)]^{(s)} = \sum_{i=q-n}^{q} B_{ij} g_i^n(x). \tag{18}$$

Note that B is an $n + 1$ by $n + s + 1$ matrix with row indices running from $q - n$ to q and column indices from $q - n - s$ to q. Below, $B[n]$ is used instead of B to indicate dependence on n.

This matrix is related to the $n+s+1$ by $n+1$ matrix D (as with B, we shall write D as $D[n]$ if necessary) specified by

$$h_i^n(t) = \sum_{j=q-n-s}^{q} D_{ji} d_j^{n+s}(t), \tag{19}$$

where the $h_i^n(t)$, $i = q-n, ..., q$, are the unique polynomials of degree n or less such that

$$(x-t)^n = \sum_{i=0}^{n} g_i(x)\, h_i(t). \tag{20}$$

These $h_i(t)$ must exist because the $g_i(x)$ are a basis for degree n or less polynomials, and $(x-t)^n$ is a degree n polynomial in x with coefficients degree n or less polynomials in t. Note that differentiating (20) with respect to x and evaluating at $x=0$ shows that $h_i(t)$ also form a basis.

Now by the same argument we used for the Pólya and generalized B-spline segments we get

$$\sum_{r=0}^{n} \frac{(-1)^{n-r}}{n!} [h_i^n(\tau)]^{(n-r)} [g_j^n(\tau)]^{(r)} = \delta_{ij}. \tag{21}$$

Sometimes it is advantageous to replace $h_i^n(t)$ by $h_i^n(t)/c_{n,i}$ for some constants $c_{n,i}$. In this case (20) and (21) become

$$(x-t)^n = \sum_{i=0}^{n} \frac{g_i(x)\, h_i(t)}{c_{n,i}} \tag{22}$$

and

$$\sum_{r=0}^{n} \frac{(-1)^{n-r}}{n!\, c_{n,i}} [h_i^n(\tau)]^{(n-r)} [g_j^n(\tau)]^{(r)} = \delta_{ij}. \tag{23}$$

Define linear functionals on the space of polynomials of degree at most n by

$$\theta_i^n f(x) = \sum_{r=0}^{n} \frac{(-1)^{n-r}}{n!\, c_{n,i}} [h_i^n(\tau)]^{(n-r)} [f(\tau)]^{(r)}. \tag{24}$$

Then by (21) the θ_i^n are the dual basis for the $g_i^n(x)$.

Also define

$$\xi_i^n f(x) = \sum_{r=0}^{n} \frac{(-1)^{n-r} [b_i^n(\tau)]^{(r)} [f(\tau)]^{(n-r)}}{n!\zeta_{n,i}}. \tag{25}$$

Note that in order for ξ_i^n to be defined it is necessary that $\zeta_{n,i} \neq 0$ for all i. Therefore for the remainder of this paper, unless otherwise stated, $\zeta_{n,i} \neq 0$ (or, equivalently, the corresponding Pólya blending functions form a basis). From Theorem 1, $\xi_i^n d_j^n = \delta_{ij}$ i.e., the ξ_i^n form the dual basis for the Pólya blending functions.

Applying θ_i^n to (18) yields

$$B_{ij} = \sum_{r=0}^{n} \frac{(-1)^{n-r}}{n!\, c_{n,i}} [h_i^n(\tau)]^{(n-r)} [b_j^{n+s}(\tau)]^{(r+s)}. \tag{26}$$

Applying ξ_j^{n+s} to (19) yields

$$\begin{aligned} D_{ji} &= \sum_{r=0}^{n+s} \frac{(-1)^{n-s-r}}{(n+s)!\, \zeta_{n+s,j}} [h_i^n(\tau)]^{(n+s-r)} [b_j^{n+s}(\tau)]^{(r)} \\ &= \sum_{r=0}^{n} \frac{(-1)^{n-r}}{(n+s)!\, \zeta_{n+s,j}} [h_i^n(\tau)]^{(n-r)} [b_j^{n+s}(\tau)]^{(r+s)}. \end{aligned} \tag{27}$$

Comparing (26) and (27) provides the relationship

$$B_{ij} = \frac{(n+s)!\, \zeta_{n+s,j}}{n!\, c_{n,i}} D_{ji}. \tag{28}$$

By interchange of the roles of the Pólya and generalized B-spline segment blending functions and using techniques similar to those employed above, it follows that

$$\hat{B}_{ij} = \frac{n!\, c_{n,j}(-1)^s}{(n+s)!\, \zeta_{n+s,i}} \hat{D}_{ji}, \tag{29}$$

where $\hat{B}$ and $\hat{D}$ are the matrices such that

$$g_j^n(x) = \sum_{i=q-n-s}^{q} \hat{B}_{ij} b_i^{n+s}(x), \qquad [d_i^{n+s}(t)]^{(s)} = \sum_{j=q-n}^{q} \hat{D}_{ji} h_j^n(t). \tag{30}$$

Some dual properties that spring from (28) and (29) are now considered.

To begin, take the case $s = 1$, $g_i^n(x) = b_i^n(x)$. Then if $c_{n,i} = \zeta_{n,i}$, it follows that $h_i^n(t) = d_i^n(t)$. This case applied to (28) therefore implies that the simple two-term differentiation formula for B-splines [5]

$$[b_j^{n+1}(x)]' = (n+1)\left[\frac{b_j^n(x)}{t_{n+j+1} - t_j} - \frac{b_{j+1}^n(x)}{t_{n+j+2} - t_{j+1}}\right] \tag{31}$$

can be derived from the the simple two-term degree elevation formula (11) for Pólya blending functions, and vice versa. Further (29) implies that degree elevation of generalized B-spline segments (a different process than degree elevation for B-spline curves) is related to differentiation of Pólya curves; while neither of these processes is simple, the main point here is that an algorithm for one will provide an algorithm for the other.

In the remainder of the cases, let $s = 0$.

Consider now a change of basis to or from the power basis. That is, let $g^n_{q-n+i}(x) = x^i$. It is not difficult to verify that

$$\sum_{r=0}^{n} \frac{(-1)^{n-r}}{n!\,(-1)^{n-j}\binom{n}{j}^{-1}} [\tau^{n-j}]^{(n-r)} [\tau^i]^{(r)} = \delta_{ij}. \tag{32}$$

Thus if $c_{n,q-n+i} = (-1)^{n-i}\binom{n}{i}^{-1}$, then $h^n_{q-n+i}(t) = t^{n-i}$. Therefore transforming generalized B-spline segments to and from the power basis is dual to transforming Pólya curves *from* and *to* the power basis.

As an example in this context, in [1, p. 46] a recursion formula for the matrix $D[n]$ transforming the Pólya basis to the power basis is given by

$$\begin{aligned} D_{ji}[n] &= \frac{\zeta_{n-1,j}\, t_j}{\zeta_{n,j}(t_{n+j} - t_j)} D_{j,i+1}[n-1] \\ &\quad - \frac{\zeta_{n-1,j+1}\, t_{n+j+1}}{\zeta_{n,j}(t_{n+j+1} - t_{j+1})} D_{j+1,i+1}[n-1] \qquad q-n \leqslant i \leqslant q-1 \\ D_{jq}[n] &= 1. \end{aligned} \tag{33}$$

Therefore, using (28), we get that the matrices given by

$$\begin{aligned} B_{ij}[n] &= \frac{n}{q-i}\left[\frac{t_{n+j+1}}{t_{n+j+1} - t_{j+1}} B_{i+1,j+1}[n-1] \right. \\ &\qquad \left. - \frac{t_j}{t_{n+j} - t_j} B_{i+1,j}[n-1]\right] \qquad q-n \leqslant i \leqslant q-1 \\ B[n]_{qj} &= 1 \end{aligned} \tag{34}$$

transform the power basis to the generalized B-spline segment basis.

Note that the matrices for transforming to the power basis (from both generalized B-spline segment blending functions and Pólya blending functions) can also be found explicitly by Marsden's Lemma.

The Bernstein basis functions satisfy the relation

$$\sum_{r=0}^{n} \frac{(-1)^{(n-r)}\left[\binom{n}{n-j}\tau^{n-j}(1-\tau)^j\right]^{(n-r)}\left[\binom{n}{i}\tau^i(1-\tau)^{n-i}\right]^{(r)}}{n!(-1)^{n-j}\binom{n}{n-j}} = \delta_{ij}, \tag{35}$$

so transforming generalized B-spline segments to and from Bézier curves is dual to transforming Pólya curves from and to Bézier curves.

Formula (35) is a consequence of Theorem 1 because if $t_{q-n+1} = \cdots = t_q = 0$ and $t_{q+1} = \cdots = t_{q+n} = 1$, then the B-spline segment curve scheme and the Pólya curve scheme coincide, and are actually just the Bézier curve scheme. Therefore, the formulas for generalized B-spline segments and Pólya curves can be particularized to results about Bézier curves.

Next suppose that the $g_i^n(x)$ are generalized B-spline blending functions over another knot vector $\hat{\mathbf{t}}$. Then the $h_i^n(t)$ are the Pólya blending functions over $\hat{\mathbf{t}}$. Thus, transforming generalized B-spline segment blending functions over the knot vector $\mathbf{t}$ to generalized B-spline segment blending functions over the knot vector $\hat{\mathbf{t}}$ is equivalent to transforming Pólya blending functions over $\hat{\mathbf{t}}$ to ones over $\mathbf{t}$. More specifically, it follows from (28) that

$$B_{ij} = \frac{\hat{\zeta}_{n,j}}{\hat{\zeta}_{n,i}} D_{ji}. \tag{36}$$

This, also, has many interesting special cases.

By (13), subdividing a Pólya curve by reparametrizing it is equivalent to transforming it to another type of Pólya curve [1, 3]. For example, to subdivide between $t=0$ and $t=\alpha$, it is necessary to find the matrix $S(\alpha)$ such that $d_i^n(\alpha t) = \sum_j S(\alpha)_{ji}\, d_j^n(t)$. From $S(\alpha)$ and (36) one can obtain the matrix which will transform the $b_i^n(\alpha x)$ to the $b_i^n(x)$. Therefore the matrix $S(1/\alpha)$ provides the subdivision matrix for generalized B-spline segments. (Note that subdivision by reparametrization is different from the usual B-spline technique of subdivision by knot insertion). Subdivision will be discussed further in the next section.

Now define the knot vector $\hat{\mathbf{t}}[m]$ by

$$\begin{aligned} \hat{t}_i &= t_i & i &\leqslant m \\ &= \hat{t} & i &= m+1 \\ &= t_{i-1} & i &> m+1. \end{aligned} \tag{37}$$

That is, insert $\hat{t}$ as the $(m+1)$st knot.

In the case that $m < q$ note that since in definitions (4) and (5) the curves are "centered" on the interval $[t_q, t_{q+1}) = [\hat{t}_{q+1}, \hat{t}_{q+2})$, it is only necessary to consider the blending functions indexed from $q-n+1$ to $q+1$; the curves will depend on the knots $t_{q-n+2}, \ldots, t_m, \hat{t}, t_{m+1}, \ldots, t_{q+n}$. The matrix indices, etc., should be adjusted accordingly. In the case where $m \geqslant q$ the blending functions will be indexed as originally and the curve will depend on the knots $t_{q-n+1}, \ldots, t_m, \hat{t}, t_{m+1}, \ldots, t_{q+n-1}$. Now by using the degree elevation formula (11), it is easy to prove that the matrix D transforming the $\mathbf{t}$ Pólya blending functions to the $\hat{\mathbf{t}}[m]$ Pólya blending functions is of the form

$$
\begin{array}{lll}
D_{j-1,j}=1 & m<j & \\[2ex]
D_{j-1,j}=\dfrac{\hat{\zeta}_{n,j}}{\check{\zeta}_{n,j-1}}\left(\dfrac{t_{j+n}-\hat{t}}{t_{j+n}-t_j}\right) & m-n\leqslant j\leqslant m & \\[2ex]
D_{j,j}=\dfrac{\hat{\zeta}_{n,j}}{\check{\zeta}_{n,j}}\left(\dfrac{\hat{t}-t_j}{t_{j+n}-t_j}\right) & m-n\leqslant j\leqslant m & \\[2ex]
D_{jj}=1 & j<m-n & \\[2ex]
D_{ij}=0 & \text{else.} & (38)
\end{array}
$$

For if $m<j$, then $d_j^n(t)=\hat{d}_{j+1}^n(t)$ from (10). If $j<m-n$, then $d_j^n(t)=\hat{d}_j^n(t)$ also by (10). If $m\geqslant j\geqslant m-n-1$, then

$$
\begin{aligned}
\hat{d}_j^n(t) &= \hat{\zeta}_{n,j}(\hat{t}-t)\prod_{i=j+1}^{j+n-1}(t_i-t)\\
&= \hat{\zeta}_{n,j}((\hat{t}-t_{j+n})+(t_{j+n}-t))\prod_{i=j+1}^{j+n-1}(t_i-t)\\
&= \frac{\hat{\zeta}_{n,j}}{\check{\zeta}_{n,j}}\,d_j^n(t)+\frac{\hat{\zeta}_{n,j}}{\check{\zeta}_{n-1,j}}(\hat{t}-t_{j+n})\,d_j^{n-1}(t)\\
&= \frac{\hat{\zeta}_{n,j}}{\check{\zeta}_{n,j}}\,d_j^n(t)+\frac{\hat{\zeta}_{n,j}}{\check{\zeta}_{n-1,j}}(\hat{t}-t_{j+n})\left[\frac{\check{\zeta}_{n-1,j}}{t_{n+j}-t_j}\left(\frac{d_j^n(t)}{\check{\zeta}_{n,j}}-\frac{d_{j-1}^n(t)}{\check{\zeta}_{n,j-1}}\right)\right]\\
&= \frac{\hat{\zeta}_{n,j}}{t_{j+n}-t_j}\left[\frac{\hat{t}-t_j}{\check{\zeta}_{n,j}}\,d_j^n(t)+\frac{t_{j+n}-\hat{t}}{\check{\zeta}_{n,j-1}}\,d_{j-1}^n(t)\right].
\end{aligned}
$$

Since the matrix D expresses the functions $\hat{d}_j^n(t)$ in terms of the functions $d_i^n(t)$, it tells us how to delete the knot $\hat{t}$ from a Pólya segment. The dual process for generalized *B*-spline segments is, of course, Boehm's knot insertion algorithm [4] for *B*-splines in this context. By (36) and (38) the insertion matrices are

$$
\begin{array}{lll}
B_{j,j-1}=1 & m<j & \\[2ex]
B_{j,j-1}=\dfrac{t_{j+n}-\hat{t}}{t_{j+n}-t_j} & m-n\leqslant j\leqslant m & \\[2ex]
B_{j,j}=\dfrac{\hat{t}-t_j}{t_{j+n}-t_j} & m-n\leqslant j\leqslant m & \\[2ex]
B_{jj}=1 & j<m-n & \\[2ex]
B_{ij}=0 & \text{else.} & (39)
\end{array}
$$

Since it is not required that the knots be non-decreasing, the knot insertion here does not depend on how the knots are ordered in value, but rather on how they are arranged in the knot vector. The processes of knot deletion for Pólya curves and knot insertion for generalized B-spline segments will be used in the next section.

In [1, p. 41] a method is presented to transform one type of Pólya curve to another. This method involves finding the transformation matrix $D[n]$ recursively using a two-term recurrence relationship

$$D[n]_{ji} = \frac{\hat{\zeta}_{n,i}}{\hat{\zeta}_{n-1,i}\zeta_{n,j}} \left[\frac{\zeta_{n-1,j}(\hat{t}_{i+n} - t_j)}{t_{j+n} - t_j} D[n-1]_{ji} + \frac{\zeta_{n-1,j+1}(t_{j+n+1} - \hat{t}_{i+n})}{t_{j+n+1} - t_{j+1}} D[n-1]_{j+1,i} \right] \tag{40}$$

for $j < q$ (there are various methods for finding the last (i.e., qth) column [1]).

Applying (36) provides the two term recurrence relationship for the associated transformation of generalized B-spline segments

$$B[n]_{ij} = \frac{\hat{t}_{i+n} - t_j}{t_{j+n} - t_j} B[n-1]_{ij} + \frac{t_{j+n+1} - \hat{t}_{i+n}}{t_{j+n+1} - t_{j+1}} B[n-1]_{i,j+1} \tag{41}$$

for $j < q$. This is the recurrence relation for the Oslo algorithm [8].

There is, in [1], another two-term recurrence relationship for Pólya curves similar in form to (40),

$$D[n]_{ji} = \frac{\hat{\zeta}_{n,i}}{\hat{\zeta}_{n-1,i+1}\zeta_{n,j}} \left[\frac{\zeta_{n-1,j}(\hat{t}_{i+1} - t_j)}{t_{j+n} - t_j} D[n-1]_{j,i+1} + \frac{\zeta_{n-1,j+1}(t_{j+n+1} - \hat{t}_{i+1})}{t_{j+n+1} - t_{j+1}} D[n-1]_{j+1,i+1} \right] \tag{42}$$

for $j > q - n$. Substituting (36) into (42) produces another Oslo-type recurrence relationship for generalized B-spline segments,

$$B[n]_{ij} = \frac{\hat{t}_{i+1} - t_j}{t_{j+n} - t_j} B[n-1]_{i+1,j} + \frac{t_{j+n+1} - \hat{t}_{i+1}}{t_{j+n+1} - t_{j+1}} B[n-1]_{i+1,j+1} \tag{43}$$

for $j > q - n$.

For examples of other change of basis formulas involving Pólya curves and generalized B-spline segments, see [1].

In summary, the dual properties of Pólya curves and generalized B-spline segments include the following:

Properties of generalized *B*-spline segments	Properties of Pólya curves
1. $[b_i^n(x)]^{(n)} = n!\zeta_{n,i}$	$[d_i^n(t)]^{(n)} = n!(-1)^n\zeta_{n,i}$
2. each $b_i^n(x)$ is of exact degree n	nondegenerate
3. $b_i^n(x) = \zeta_{n,i}(x - \hat{i})^n$	interpolates P_i at $\hat{i}$
3a. $b_i^n(x) = \zeta_{n,i}(x - s_{i+n-q})^n$ $i = q-n, \ldots, q$	the $d_i^n(t)$ are the Lagrange cardinal functions
4. interpolates P_{q-n} [resp. P_q] at t_q [t_{q+1}]	$d_{q-n}^n(t) = \zeta_{n,q-n}(t_q - t)^n$ [$d_q^n(t) = \zeta_{n,q}(t_{q+1} - t)^n$]
5. degree elevation	differentiation
6. differentiation	degree elevation
7. conversion to power basis by Marsden's Lemma	conversion to power basis by Marsden's Lemma
8. change of basis to [from] *h*-curves	change of basis from [to] *g*-curves
8a. change of basis to [from] power basis	change of basis from [to] power basis
8b. change of basis to [from] Bézier	change of basis from [to] Bézier
8c. subdivision (by reparametrization)	subdivision (by reparametrization)
8d. knot insertion	knot deletion
8e. Oslo algorithm	two-term recurrence for transforming one type of Pólya curve to another
8f. other Oslo-type result	other two-term recurrence for transforming one type of Pólya curve to another
9. arise from opposite color urn model	arise from same color urn model

5. Some New Results for Pólya Curves and Generalized *B*-Spline Segments

In this section we will use the results of the previous section, in particular the knot insertion/knot deletion techniques, to prove some new results about Pólya curves and generalized *B*-spline segments.

The underlying idea in these results is that repeated knot deletion [insertion] can be used to change one type of Pólya curve [generalized *B*-spline segment] to another. Suppose we wish to change a $\hat{\mathbf{t}}$ Pólya curve to a $\mathbf{t}$ Pólya curve. A strategy for doing so is

Algorithm 1. Steps $k = 1, \ldots, n$: delete the $q - n + k$th $\hat{\mathbf{t}}$ knot and introduce as the new "first" knot the value t_{q+1-k}.

Steps $k = n+1, \ldots, 2n$: delete the $q + 2n + 1 - k$th $\hat{\mathbf{t}}$ knot and introduce as the new "last" knot the value t_{q-n+k}.

To transform a $\mathbf{t}$ generalized *B*-spline segment curve to a $\hat{\mathbf{t}}$ generalized *B*-spline segment curve we use knot insertion:

ALGORITHM 2. Steps $k = 1, ..., n$: insert $\hat{t}_{q+1-k}$ as the new $q+1-k$th knot (thereby making the curve no longer dependent on t_{q-n+k}).

Steps $k = n+1, ..., 2n$: insert $\hat{t}_{q-n+k}$ as the new $q-n+k$th knot (thereby making the curve no longer dependent on $t_{q+2n+1-k}$).

Therefore the transformation matrices for transforming one type of Pólya curve to another or transforming one type of generalized B-spline segment to another are products of at most $2n$ knot deletion or knot insertion matrices, whose forms are given in (38) and (39). The matrices transforming the blending functions are the same ones which transform the control points [9]. Therefore there are simple algorithms for these transformations. In particular, they consist of at most $2n$ steps; in the kth step new control points $Q_j^{[k]}$ are obtained from a linear combination of either the odd control points $Q_j^{[k-1]}$ and $Q_{j+1}^{[k-1]}$ or the old control points $Q_{j-1}^{[k-1]}$ and $Q_j^{[k-1]}$ (where the $Q_j^{[0]}$ are the original control points, and the $Q_j^{[2n]}$ will be the new control points). Further details are given in [1, 3].

Having briefly presented these algorithms, we will now use the techniques of knot insertion and deletion to prove some results about generalized B-spline segments and Pólya curves. Recall from Section 3 that the $b_i^n(x)$ always form a basis for the space of polynomials of degree at most n. Here is another result concerning the $b_i^n(x)$.

THEOREM 5. *Let $a = \max_{i=q-n+1,...,q} t_i$ and $b = \min_{i=q+1,...,q+n} t_i$. If $a < b$, then the $b_i^n(t)$ satisfy Descartes' Law of Signs on (a, b).*

Proof. Use Algorithm 2, inserting a in steps 1 through n and b in steps $n+1$ through $2n$. Then the resulting blending functions are reparametrized Bernstein basis functions. These satisfy Descartes' Law of Signs on (a, b). Moreover, the product of the insertion matrices transforms the reparametrized Bernstein basis functions into the original $b_i^n(x)$. Now if this product is strictly totally positive, the original $b_i^n(x)$ satisfy Descartes' Law of Signs on (a, b) [9]. To prove that the product is strictly totally positive, it is sufficient to prove that each insertion matrix is strictly totally positive. This, in turn, follows directly from the form of the insertion matrices given in (39).

COROLLARY. $\mathbf{b}_q(t)$ is variation diminishing on $[a, b]$.

Proof. This follows from Theorem 5 and the discussion of Descartes' Law of Signs in [9]. Q.E.D.

Note that since we are in a different setting the corollary is not merely a restatement of the variation diminishing property of B-spline curves.

The last topic we will discuss for generalized B-spline segments is subdivision by reparametrization. A reparametrization result for the $b_i^n(x)$

similar to (13) holds [1, p. 96]: let $\hat{b}_i^n(x)$ denote the generalized B-spline segment blending functions dependent on the knot vector

$$\hat{\mathbf{t}} = \left(\frac{t_{q-n+1}-c}{a}, \ldots, \frac{t_q - c}{a}, \frac{t_{q+1}-c}{a}, \ldots, \frac{t_{q+n}-c}{a}\right);$$

then

$$b_i^n(ax+c) = \hat{b}_i^n(x). \tag{44}$$

Now suppose we have a generalized B-spline curve over a finite interval, and wish to express a portion of that curve as the same type of generalized B-spline curve. That is, we have two curves, the original one and the portion of it. Both can be represented as generalized B-spline segments with the same knots and domain; only the control points will differ. (Note this type of subdivision differs from usual B-spline subdivision by knot insertion, although it is similar to Bézier curve subdivision.) This process can be done by using the reparametrization result above and then doing a change of basis between the $\hat{b}_i^n(x)$ and the $b_i^n(x)$ with Algorithm 2.

We now turn our attention to Pólya curves. Algorithm 1 leads to many results for Pólya curves. We sketch some of these here; a more detailed discussion is the topic of [3].

THEOREM 6. *Let* $a = \max_{i=q-n+1,\ldots,q} \hat{t}_i$ *and* $b = \min_{i=q+1,\ldots,q+n} \hat{t}_i$. *If* $a < b$, *then the* $\hat{d}_i^n(t)$ *satisfy Descartes' Law of Signs on* (a, b).

Proof. Use Algorithm 1, introducing the value a in steps 1 through n and b in steps $n+1$ through $2n$. Then the resulting blending functions are reparametrized Bernstein basis fuctions, and the product of the deletion matrices transforms the reparametrized Bernstein basis fuctions into the original $\hat{d}_i^n(x)$. Thus, as in the proof of Theorem 5, it is sufficient to prove that each deletion matrix is strictly totally positive. This will follow from the form of the deletion matrices given in (38) and the following observations: when a knot is deleted in steps 1 through n, all the knots with smaller subscripts have value a. When a knot is deleted in steps $n+1$ through $2n$ all the knots with larger subscripts have value b. In steps 1 through n the ratio of the ζ's appearing in the expression for $D_{j-1,j}$ in (38) is positive and the ratio appearing in the expression for D_{jj} in (38) is negative [1, 3] (remember the remarks concerning the indices in the paragraph above Eq. (38)). In steps $n+1$ to $2n$ the ratio of the ζ's appearing in the expression for $D_{j-1,j}$ in (38) is negative and the ratio appearing in the expression for D_{jj} is positive [1, 3].

COROLLARY. $\hat{\mathbf{d}}_q(t)$ *is variation diminishing on* $[a, b]$.

One thing lacking in previous discussions of Pólya curves [1, 2] was a simple subdivision algorithm. However, Eq. (13) and Algorithm 1 furnish a subdivision technique analogous to the technique just described for generalized B-spline segments, although the presence of the ζ's makes the algorithm for Pólya curves somewhat more complicated.

Our final result concerns the fact that Pólya curves interpolate their first and last control points. Knot removal, therefore, gives us an evaluation algorithm. In order to evaluate a Pólya curve at s, use steps 1 through n to get the new qth knot equal to s, or steps $n+1$ through $2n$ to get the new $q+1$st knot equal to s. In either case the deletion matrices can then be applied to the original control points to get a new set of control points, the first or last control point of which will be the desired value. Like the subdivision algorithm, this algorithm is complicated by the presence of the ζ's [1, 3].

Note that the evaluation of generalized B-spline curves can be done via n-fold knot insertion (essentially the de Boor algorithm) and evaluation of Pólya curves via n-fold knot deletion.

6. Concluding Remarks

We have examined the relationship between two curve schemes, generalized B-spline segments and Pólya curves, and have shown that the duality provided by the (modified) de Boor–Fix form of the dual basis connects certain results for these two schemes. Therefore these two curve schemes can be considered as dual in this sense. Note that many of the results mentioned above do not depend on particular properties of generalized B-spline segments of Pólya curves, but work for any dual schemes.

Some open questions still remain. As mentioned in Section 2, there are probabilistic models from which we can derive generalized B-spline segments and Pólya curves. Many properties of these curve schemes can be obtained by probabilistic considerations [1, 10, 11]. Are there probabilistic proofs or interpretations for any of the results contained in this paper (in particular for Marsden's Lemma and the de Boor–Fix formula), or does probability theory in any way provide insight into them? Also, can the results here be extended to surfaces in any way? Finally, since many of these results are true for any dual schemes, are there any other dual schemes that are of interest either in approximation theory or in CAGD?

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